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Enhancement of Turbulence in a Stratified Fluid by the Presence of a Shear Field

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Time-dependent solutions are obtained for turbulent flow in a stratified fluid in the presence of a shear field. Within the stated closure assumptions, it is shown that for certain shear fields that the turbulent intensity is enhanced.

KEY WORDS: Fluctuating velocity field; Reynolds stress; shear field; stratified fluid; time-dependent solutions; turbulent intensity; critical flux Richardson number.

The basic aspects of turbulent–shear interaction are clearly identified by considering the evolution of each of the components of the Reynolds stress tensor.[1-3] The x axis is taken along the direction of the mean (shear) flow $U(z)$ and we neglect variations in the mean and turbulent flow in the horizontal plane. (Coupling between the vertical and horizontal is considered through pressure fluctuations.) In addition, neglecting terms describing the variable mass of the fluid and those describing redistribution of the Reynolds stress component in space results in

$$-\frac{\partial \overline{u^2}}{\partial t} = \frac{2}{\rho}\overline{u\frac{\partial p}{\partial x}} + 2\overline{uw}\frac{dU}{dz} + 2\epsilon/3 \tag{1}$$

$$-\frac{\partial \overline{v^2}}{\partial t} = \frac{2}{\rho}\overline{v\frac{\partial p}{\partial y}} + \frac{2\epsilon}{3} \tag{2}$$

$$-\frac{\partial \overline{w^2}}{\partial t} = \frac{2}{\rho}\overline{w\frac{\partial p}{\partial z}} + 2\rho g\overline{\zeta w} + \frac{2\epsilon}{3} \tag{3}$$

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$$-\frac{\partial \overline{uw}}{\partial t} = \frac{1}{\rho}\left(\overline{u\frac{\partial p}{\partial z}} + \overline{w\frac{\partial p}{\partial z}}\right) + \rho g\overline{\zeta u} + \overline{w^2}\frac{dU}{dz} + \text{molecular terms} \quad (4)$$

$$-\frac{\partial \overline{\zeta w}}{\partial t} = \frac{1}{\rho}\overline{\zeta\frac{\partial p}{\partial z}} + \rho g\overline{\zeta^2} + \overline{w^2}\frac{d(1/\rho)}{dz} + \text{molecular terms} \quad (5)$$

$$-\frac{\partial \overline{\zeta^2}}{\partial t} = \overline{\zeta w}\frac{d(1/\rho)}{dz} + \text{molecular terms} \quad (6)$$

where u, v, and w are the three fluctuating velocity components in the x, y, and z directions, respectively, ρ is the mean density, P is the mean pressure, ζ is the fluctuating part of the specific volume, p is the fluctuating component of the pressure, and the molecular terms include the viscous dissipative effects. In Eqs. (1)–(3) the viscous terms have been modeled as those appropriate to isotropic turbulence. The shear–turbulence interaction is included in Eqs. (1) and (4) through the term $2\overline{uw}(dU/dz)$ and $\overline{w^2}(dU/dz)$, respectively. $2\overline{uw}(dU/dz)$ describes the coupling of the $\overline{uw}$ component of the Reynolds stress as it couples to the shear dU/dz. It is this coupling which feeds energy into the $\overline{u^2}$ term of the turbulent intensity. In a similar fashion, the $\overline{w^2}$ (dU/dz) term contributes to the $\overline{uw}$ component of the Reynolds stress. Thus when a vertical component of the turbulence field, $\overline{w^2}$, is nonzero the shear–turbulence interaction feeds the $\overline{uw}$ component of the Reynolds stress. This enhanced $\overline{uw}$ component then leads to an increase in $\overline{u^2}$ through the $2\overline{uw}$ (dU/dz) coupling in Eq. (1). Thus the energy in the shear field directly enhances $\overline{u^2}$, which is redistributed to $\overline{v^2}$ and $\overline{w^2}$ through the agency of pressure fluctuations. Thus, the shear–turbulence interaction can transfer energy from the shear field to the turbulent field and lead to the enhancement of the turbulent intensity. In particular, the vertical component of the Reynolds stress, $\overline{w^2}$, need not decay faster than the horizontal components of the turbulence.

The balance of the turbulent intensity is investigated by summing Eqs. (1)–(3). In this summation the pressure fluctuation terms[2] are omitted since they only redistribute energy in physical space. Thus we have

$$\frac{\partial b}{\partial t} = -\overline{uw}\frac{dU}{dz} - \rho g\overline{\zeta w} - \epsilon \quad (7)$$

where $b = (1/2)\ (\overline{u^2} + \overline{v^2} + \overline{w^2})$ is the turbulent intensity. The first term on the right-hand side represents the rate of energy transfer per unit volume from the shear field to the turbulent field. The second term defines the rate of energy loss per unit volume to the buoyancy in the stably stratified fluid, and the third term represents the usual dissipation rate. The ratio of the rate of energy loss to buoyancy to the total power passing through the turbulence in steady state is known as the flux Richardson number, Rf. Thus, Eq. (7) becomes

$$\frac{\partial b}{\partial t} = -\overline{uw}\,\frac{dU}{dz}(1 - \mathrm{Rf}) - \epsilon \tag{8}$$

where Rf is understood to be below the critical value of the flux Richardson number. If the energy loss to the buoyancy exceeds the total energy input then the turbulence will not be sustained by the energy delivered to the turbulence from the shear field. Thus, the value of Rf for which turbulence can be sustained is less than 1. The critical flux Richardson number has been estimated in various experimental and theoretical treatments, it is known that this critical value is approximately 0.15 and 0.30.

We consider a solution to the energy balance equation, Eq. (8), subject to the restriction that the flux Richardson number is approximately constant. In addition, for closure, we shall assume that the $\overline{uw}$ component of the Reynolds stress is proportional to the turbulent intensity. This closure model is crude and in making a specific application of the results obtained herein it is expected that this proportionality would be verified for the data set in question.

Let us assume the experimentally determined relation[7,8]

$$\epsilon \approx (2b)^{3/2}/l \tag{9}$$

where l is the scale of the energy containing eddies. In this analysis it is convenient to characterize the fluctuations by a wavelength $\lambda = 2\pi l$. By defining $q^2 = 2b$, utilizing Eq. (9), and assuming the closure relation,

$$\overline{uw} = \delta q^2 \tag{10}$$

Equation (8) takes the form

$$\frac{dq}{dt} = \delta q\,\frac{dU}{dz}(1 - \mathrm{Rf}) - \frac{q^2}{l} \tag{11}$$

where dU/dz is taken as positive. Letting

$$\alpha f(t) = \delta\,\frac{dU}{dz}(1 - \mathrm{Rf}) \tag{12a}$$

where $f(t)$ expresses the time variation of the shear and

$$\beta = 1/l \tag{12b}$$

Eq. (11) takes the form

$$\dot{q} = \alpha f(t) q - \beta q^2 \tag{13}$$

Equation (13) cannot be integrated for general forms of $f(t)$; indeed, this is true even for the simple form $f(t) = t^{\gamma}$ when all values of γ are considered. However, the integration is possible in terms of elementary functions when $\gamma = 0$ or $\gamma = -1$ and when $\gamma = 1$ the solution of Eq. (13) contains Dawson's integral,[9] which is a tabulated function. Taking $\nu = 1/q$ results in

$$\dot{\nu} + \alpha f(t)\nu = \beta \tag{14}$$

With $f(t) = (t/\tau)^{\gamma}$ it is clear that the integrating factor for Eq. (14) is $\exp[\alpha\tau(t/\tau)^{\gamma+1}/(\gamma+1)]$. The solution of this equation can be written as a double integral by utilizing the integrating factor $\exp[\alpha \int^t f(t)\,dt]$; however, the resulting expression is impossible to integrate except for very simple values of $f(t)$. Thus, for $\gamma \neq -1$,

$$\frac{d}{dt}\left[\nu \exp\left(\frac{\alpha}{\gamma+1}\,\frac{t^{\gamma+1}}{\tau^{\gamma}}\right)\right] = \beta \exp\left(\frac{\alpha}{\gamma+1}\,\frac{t^{\gamma+1}}{\tau^{\gamma}}\right) \tag{15}$$

For $\gamma = 0$, the integral of Eq. (15) yields $\nu(= 1/q)$ such that

$$q(t) = \left[\frac{\beta}{\alpha}\left(1 - e^{-\alpha(t-t_0)}\right) + \frac{1}{q(t_0)}\,e^{-\alpha(t-t_0)}\right]^{-1} \tag{16}$$

where $q(t_0)$ is the initial value of $q(t)$ at the time t_0. Here we have assumed that both dU/dz and Rf are independent of time. The interaction time is crudely defined by

$$T = \frac{1}{\alpha} = \frac{1}{\delta(dU/dz)(1 - \mathrm{Rf})} \tag{17}$$

and

$$q(t) \underset{t-t_0 \gg T}{\rightarrow} \frac{\alpha}{\beta} = l\delta\frac{dU}{dz}(1 - \mathrm{Rf}) \tag{18}$$

For $\delta = 0.15$, $\mathrm{Rf}_{\mathrm{crit}} = 0.15$ and $dU/dz = 10^{-2}\ \mathrm{sec}^{-1}$, the interaction time $T = 784$ sec. Shears greater than $10^{-2}\ \mathrm{sec}^{-1}$ have been reported in oceanic environments.[10] Furthermore, for the sake of illustration we take $q(\infty) = \alpha/\beta$. This terminal value of q, $q(\infty)$ depends only on the scale of the energy-containing eddies, the constant δ, the magnitude of the shear, and the critical value of the flux Richardson number. Thus the ratio $q(t_0)/q(\infty)$ may take on many particular values. In Fig. 1 the time evolution of $q(t)/q(t_0)$ as predicted by Eq. (16) is plotted as a function of $(t - t_0)/T$ for $q(t_0)/q(\infty) = 1/10$, $1/3$, 1, 3, and 10. For a given $q(t_0)$, the ratio of $q(t_0)/q(\infty)$ decreases as l or dU/dz increases and for $q(t_0)/q(\infty) < 1$ the ratio $q(t)/q(t_0)$ increases as $(t - t_0)/T$ increases. Figure 1 illustrates that the rate of increase of $q(t)/q(t_0)$ is more rapid for larger values of l or dU/dz when $q(t_0)/q(\infty) \leqslant 1$. This increase in $q(t)/q(t_0)$ is due completely to the energy coupled to the turbulent flow from the shear. The main point

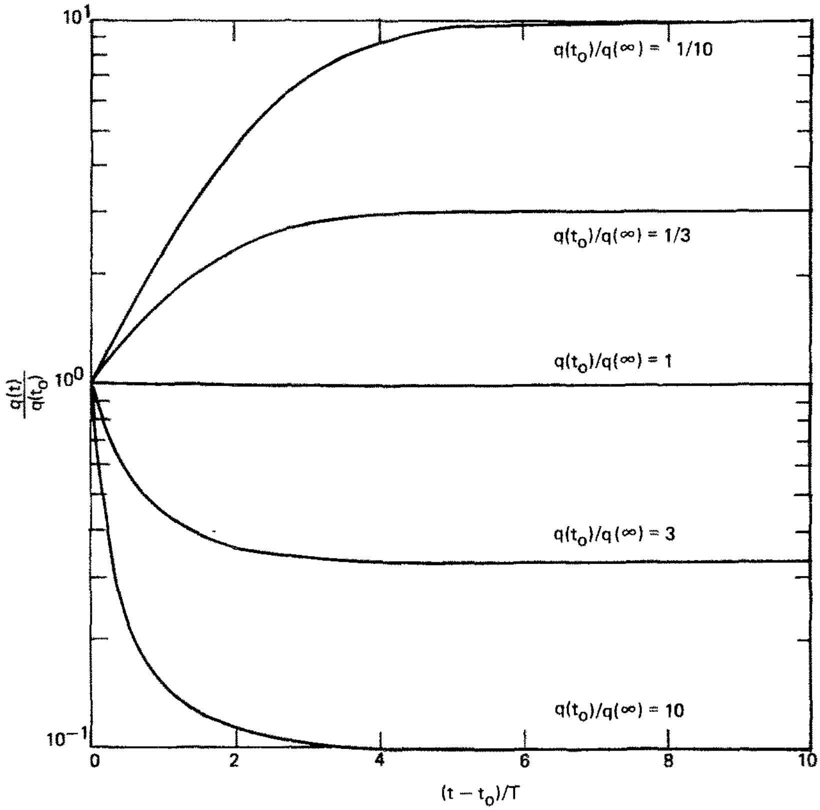

Fig. 1. $q(t)/q(t_0)$ as a function of $(t - t_0)/T$.

of Fig. 1 is that the dissipative decay of the turbulent intensity is arrested when the energy density flow due to the shear–turbulence interaction is comparable to the dissipation rate.

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